An algorithm based on graphs for solving a fair division problem.

William Olvera López   Francisco Sánchez Sánchez

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Introduction

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- **Example:** To divide a pizza.
- **Main result:** An algorithm to find the highest even solution.

Definitions and notation

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- $A = (a_{ij})_{i \in M, j \in N}$ the valuation that agent j gives to item i. We suppose

$$\sum_{i \in M} a_{ij} = c$$

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  for every $j \in N$ and some $c \in \mathbb{R}$.
- $X = (x_{ij})_{i \in M, j \in N}$ where x_{ij} is the percentage of good i assigns to agent j.

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- The problem for find the best even solution

$$\begin{aligned} \max \quad & z \\ s.a. \quad & \\ & \textstyle\sum_{i \in M} a_{ij} x_{ij} \geq z \quad j \in N \\ & \textstyle\sum_{j \in N} x_{ij} = 1 \quad i \in M \\ & x_{ij} \geq 0 \end{aligned}$$

Definitions and notation

- Dual problem

$$
\begin{aligned}
\min \quad & \textstyle\sum_{i \in M} v_i \\
a_{ij} u_j &\le v_i \quad i \in M, j \in N \\
\textstyle\sum_{j \in N} u_j &= 1 \\
u_j &\ge 0
\end{aligned}
$$

The dual variables u_j tell us how the even value z decreases if player j is given one extra unit of utility. The dual variables v_i tell us how the even value z increases if we add one unit of item i.

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- **Remark 2:** Kuhn-Tucker conditions

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- **Remark 3:** We associate a graph to every basic solution (it has no cycles).

$$g = \{(i, j) | x_{ij} > 0\}$$

Solution of the dual problem associated to the graph

- If we know the optimal graph, then the solution of the dual problem is direct. We need to solve the system

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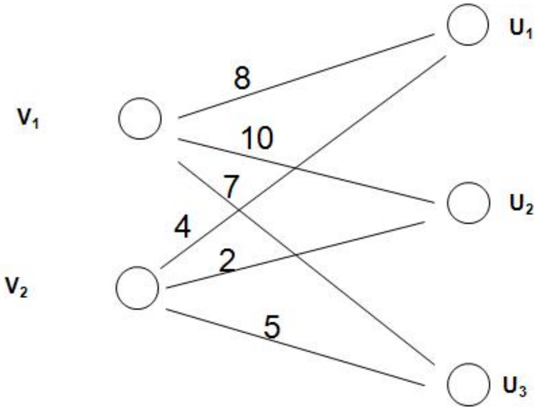
- Notice that,

$$z = \sum_{i \in M} v_i$$

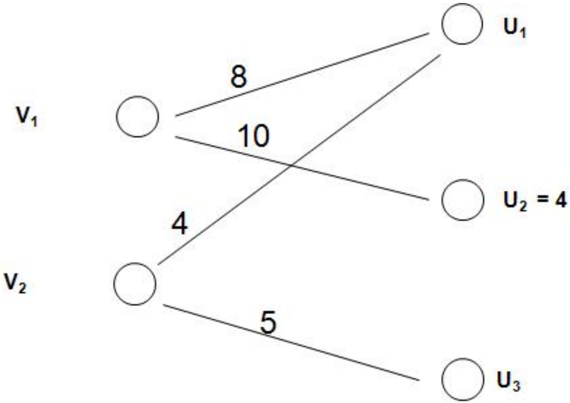
Example

$$A = \begin{bmatrix} 8 & 10 & 7 \\ 4 & 2 & 5 \end{bmatrix}$$

Example

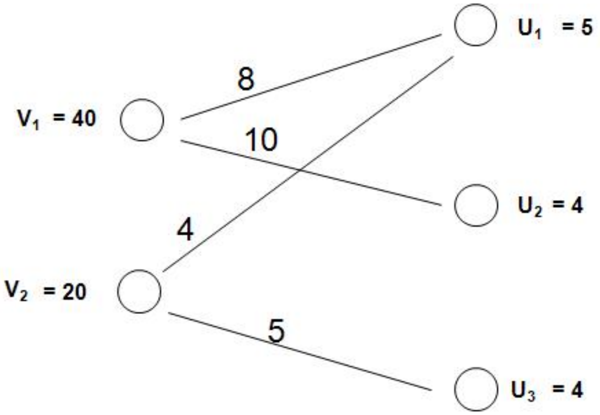


Example



we need $a_{ij} u_j = v_i$ for $(i, j) \in g$.

Example



Example

$$V_i := \frac{V_i}{U_1 + U_2 + U_3}$$

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$$\mathsf{V}= \left(\frac{40}{13}, \frac{20}{13}\right)$$

$$\mathsf{U}= \left(\frac{5}{13}, \frac{4}{13}, \frac{4}{13}\right)$$

Solution of the primal problem associated to the graph

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$$\begin{aligned} \sum_{i \in M} a_{ij} x_{ij} &= z \\ \sum_{j \in N} x_{ij} &= 1 \end{aligned}$$

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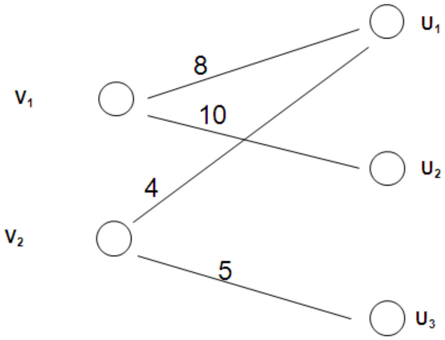
- If there is only one edge connecting $j \in N$ then $x_{ij} = \frac{z}{a_{ij}}$.
- If there is only one edge connecting $i \in M$ then $x_{ij} = 1$.

Example

- 

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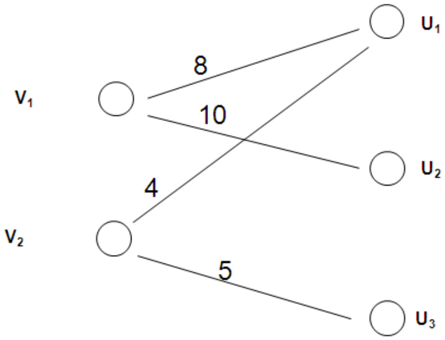
- 

- Since $z = \sum_{i \in M} v_i = \frac{60}{13}$ and $\sum_{i \in M} a_{ij} x_{ij} = z$

$$x_{23} = \frac{1}{5}\frac{60}{13} = \frac{12}{13} \quad x_{21} = 1 - x_{23} = \frac{1}{13} \quad x_{12} = \frac{6}{13} \quad x_{11} = \frac{7}{13}$$

Algorithm

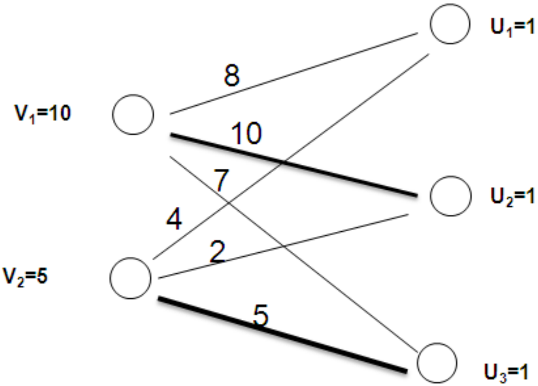
1. Find a feasible basic a-graph for A with associated dual vectors u,v.
2. Assume $z = w = \sum_{i \in M} v_i$
3. Find the value of the primal variables X. If X is a feasible primal solution, then the algorithm ends.
4. Otherwise, find a new feasible basic a-graph and return to Step 2.

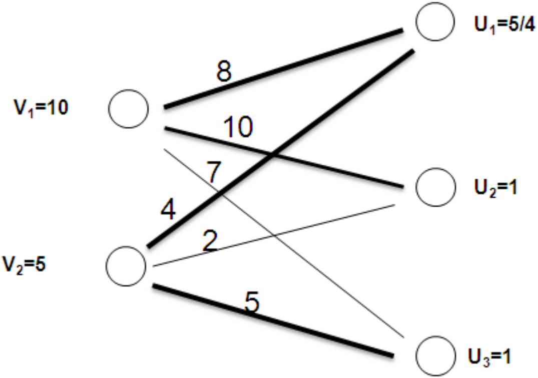
Find a feasible basic a-graph

1. Include the maximum entry for each row.
2. Include $n-1$ additional entries keeping dual feasibility

$$(h,k) := \arg\min \left\{ \frac{v_i}{a_{ij} u_j} : (i,j) \notin E, i \notin N(C_j) \right\}.$$

3. Update v and u.


V1=10
V2=5
U1=1
U2=1
U3=1
8
10
7
4
2
5



$u = (\frac{5}{13}, \frac{4}{13}, \frac{4}{13})$ and $v = (\frac{40}{13}, \frac{20}{13})$.

Finding new feasible dual vectors

1. Suppose $x_{hk} < 0$. Set $E' := E \backslash (h, k)$.
2. Calculate
$$(i, j) := \arg\min \left\{ \frac{v_i}{a_{ij} u_j} : (i, j) \notin E',\ i \in N(C_k), j \in N(C_h) \right\}.$$
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Example 2.

- $$A = \begin{bmatrix} 15 & 16 & 4 \\ 3 & 2 & 14 \end{bmatrix}$$

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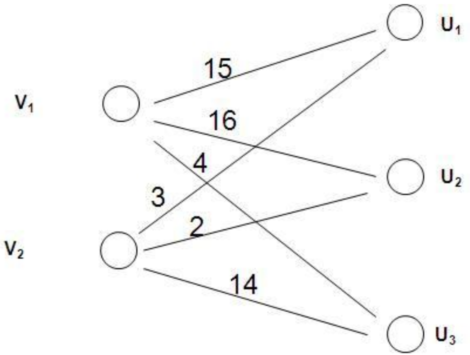
- 

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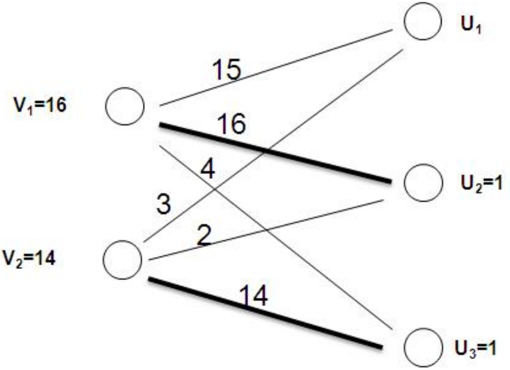

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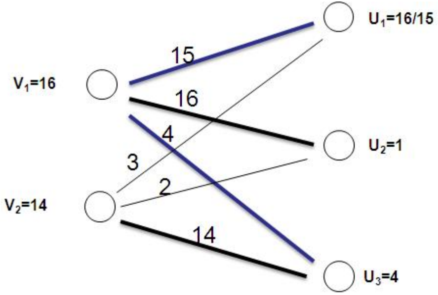
- 

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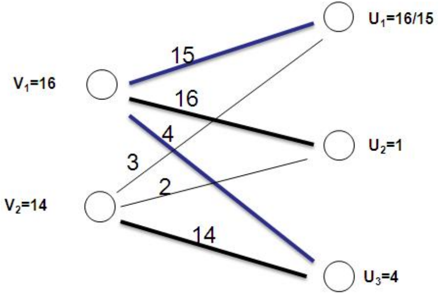
- 

- Since $\sum u_j = \frac{91}{15}$ then

$$v^T = (\frac{16(15)}{91}, \frac{14(15)}{91}) \qquad u^T = (\frac{16}{15}\frac{15}{91}, \frac{15}{91}, \frac{4(15)}{91})$$

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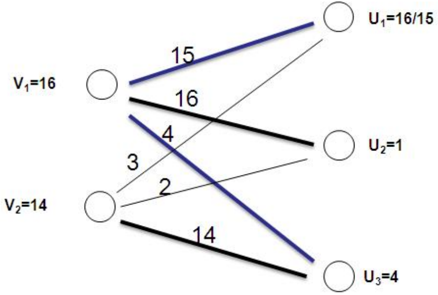
- 

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- and

$$x_{23} = 1 \quad x_{11} = \frac{72}{91} \quad x_{12} = \frac{135}{182} \quad x_{13} = -\frac{97}{182}$$

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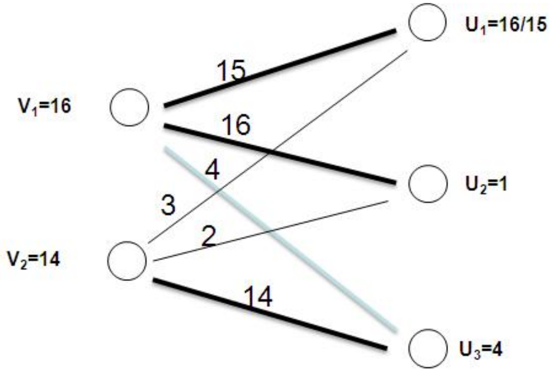
- 

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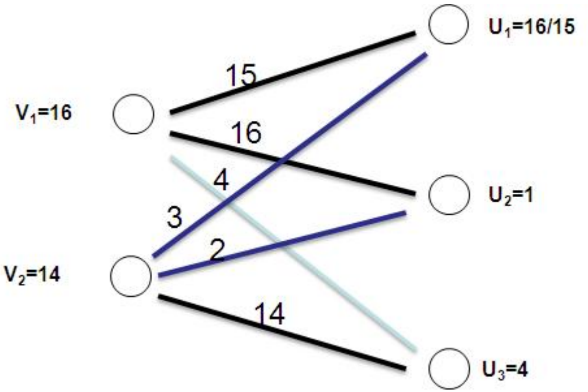

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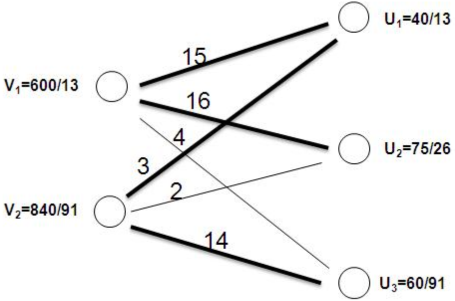
- 

Since $\sum u_j = \frac{1205}{182}$ then

$$v^T = \left(\frac{1680}{241}, \frac{336}{241}\right) \qquad u^T = \left(\frac{112}{241}, \frac{105}{241}, \frac{24}{241}\right)$$

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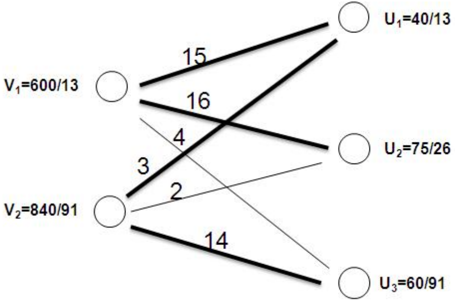
- 

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- and

$$x_{21} = \frac{97}{241} \quad x_{11} = \frac{115}{241} \quad x_{12} = \frac{126}{241} \quad x_{23} = \frac{144}{241}$$

Theorem *The previous algorithm produce the highest even solution.*

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